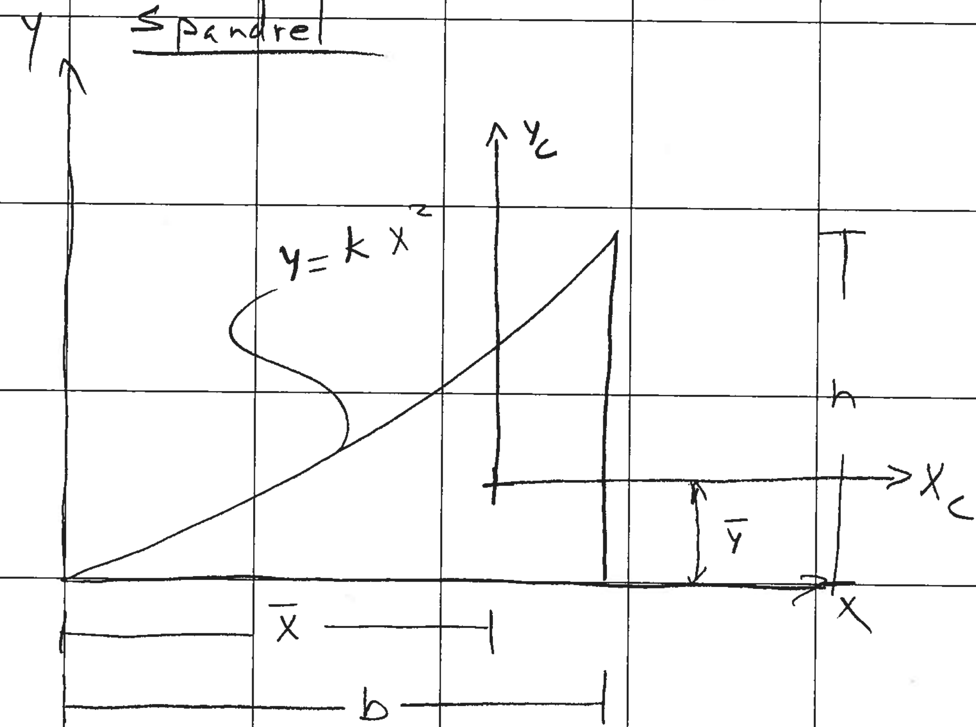
Summary of Cross-Section Properties for Parabolic Spandrel

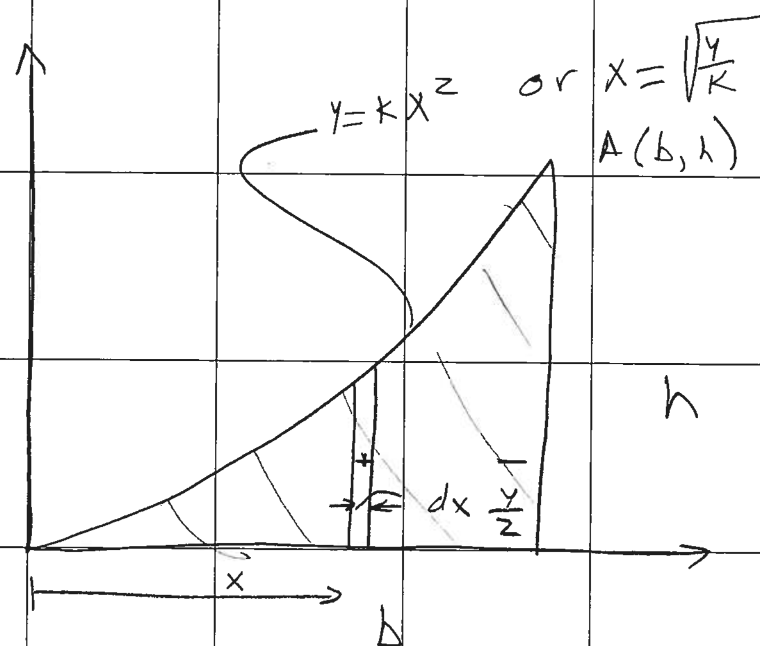


$$A = \frac{bh}{3}, \quad \bar{x} = \frac{3}{4}b, \quad \bar{y} = \frac{3}{10}h$$

$$I_x = \frac{bh^3}{21}, \quad I_y = \frac{hb^3}{5}$$

$$I_{xc} = \frac{37}{2100}bh^3, \quad I_{yc} = \frac{hb^3}{80}$$

Determine the location of the centroid for the shaded area shown



First determine the value of k given coordinates of A.

(1) $y = kx^2$

(2) $h = kb^2$

(3) $k = \frac{h}{b^2}$

So

(4) $y = \frac{h}{b^2}x^2$ or $x = b\sqrt{\frac{y}{h}}$

Calculate the area.

(5) $dA = y\,dx$

(6) $dA = \frac{h}{b^2} x^2\,dx$

(7) $A = \frac{h}{b^2} \int_0^b x^2\,dx$

(8) $A = \frac{h}{b^2} \frac{b^3}{3}$

(9) $A = \frac{bh}{3}$

Calculate $\int \tilde{y}\,dA$

$\tilde{y}$ distance from x-axis to centroid of diff. element.

(10) $\int \tilde{y}\,dA = \int_0^b \frac{1}{2} y \frac{h}{b^2} x^2\,dx$

(11) $= \int_0^b \frac{1}{2}\left(\frac{h}{b^2} x^2\right)\left(\frac{h}{b^2}\right) x^2\,dx$

(12) $= \frac{h^2}{2b^4} \int_0^b x^4\,dx$

(13) $= \frac{h^2}{2b^4} \frac{1}{5} b^5$

(14) $\int y \, dA = \dfrac{h^2 b}{10}$

Calculate $\overline{y}$

(15) $\overline{y} = \dfrac{\int y \, dA}{\int dA}$

(16) $\overline{y} = \dfrac{\frac{h^2 b}{10}}{\frac{b h}{3}}$

(17) $\overline{y} = \dfrac{3 h^2 b}{10 b h}$

(18) $\overline{y} = \dfrac{3}{10} h$

Calculate $\int x \, dA$

distance from x-axis to centroid of diff. element

(19) $\int x \, dA = \int_0^b x \dfrac{h}{b^2} x^2 \, dx$

(20) $= \dfrac{h}{b^2} \int_0^b x^3 \, dx$

(21) $= \dfrac{h}{b^2} \dfrac{1}{4} x^4 \Big|_0^b$

(22) $$\int x\, dA = \frac{h\, b^4}{4\, b^2}$$

(23) $$\int x\, dA = \frac{h b^2}{4}$$

Calculate $\bar{X}$

(24) $$\bar{X} = \frac{\int x\, dA}{\int dA}$$

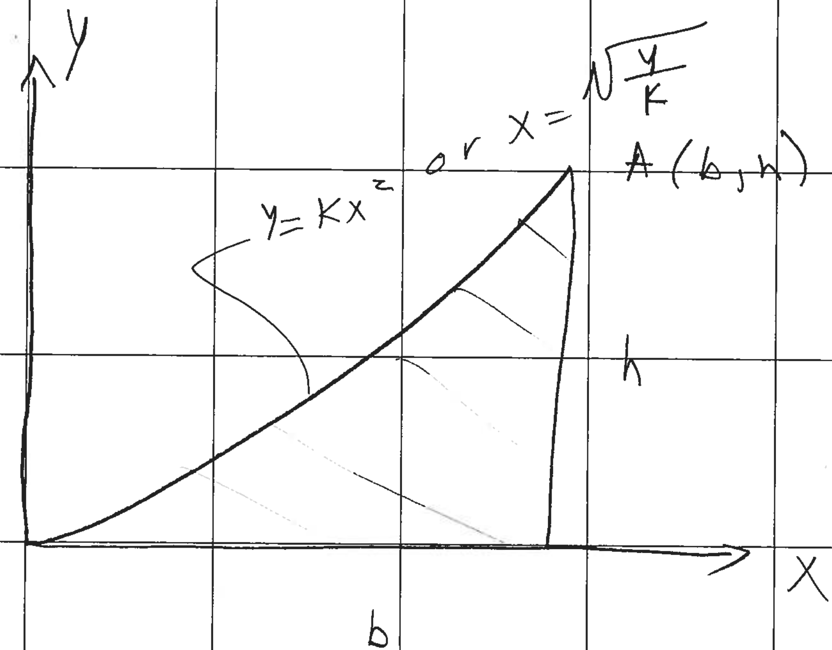
(25) $$\bar{X} = \frac{\frac{h b^2}{4}}{\frac{b h}{3}}$$

(26) $$\bar{X} = \frac{3 h b^2}{4 b h}$$

(27) $$\bar{X} = \frac{3}{4} b$$

$\bar{X} = \frac{3}{4} b$ and $\bar{y} = \frac{3}{10} h$

Calculate the moments of inertia for the shaded area shown about the X and y axes.

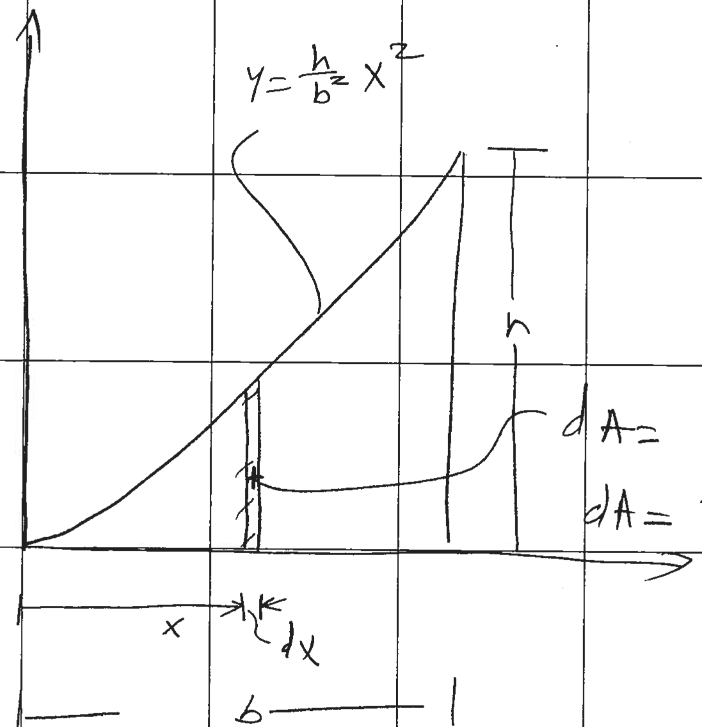


From Previous Example

(1) $$y = \frac{h}{b^2} x^2 \quad \text{or} \quad x = b\sqrt{\frac{y}{h}}$$

Calculate I_y

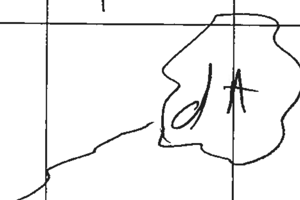
choose a differential element that is parallel to the y axis as shown



$dA = y\,dx$

$dA = \frac{h}{b^2} x^2\,dx$

(2) $I_y = \int x^2\,dA$



(3) $I_y = \int_0^b x^2 \left(\frac{h}{b^2} x^2\right) dx$

(4) $I_y = \frac{h}{b^2} \int_0^b x^4\,dx$

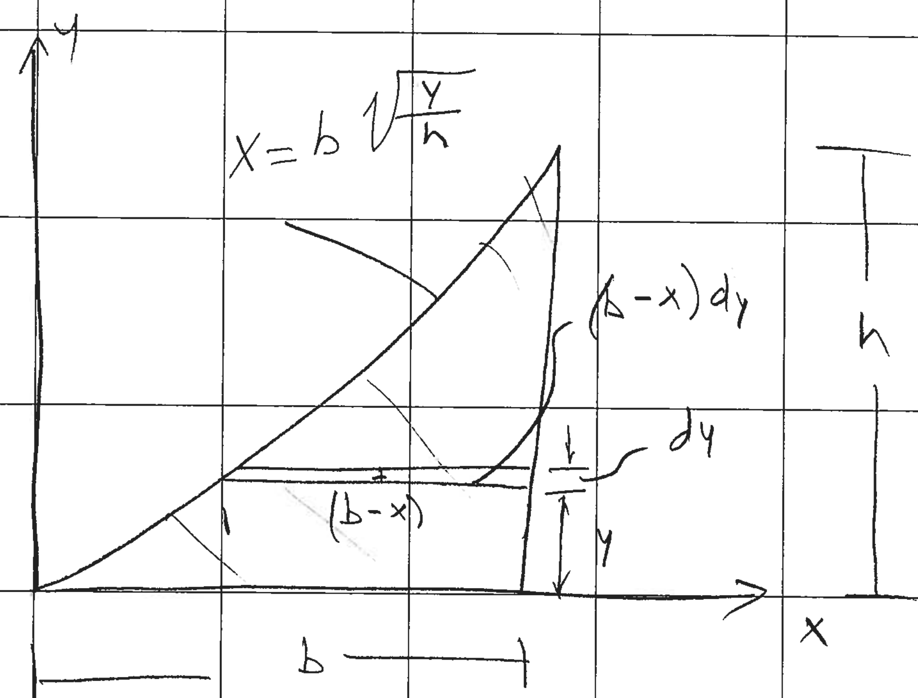
(5) $= \frac{h}{b^2} \frac{1}{5} x^5 \Big|_0^b$

(6) $I_y = \frac{h}{b^2} \frac{1}{5} b^5$

(7) $$I_y = \frac{h b^3}{5}$$

Calculate Ix

Choose a differential element that is parallel to X-axis as shown



(8) $$I_x = \int y^2 \, dA$$

(9) $$I_x = \int_0^h y^2 (b-x)\, dy$$

(10) $$I_x = \int_0^h y^2 \left(b - b\sqrt{\frac{y}{h}}\right) dy$$

(11) $$I_x = \int_0^h y^2 b\,dy - \int_0^h y^2 b\sqrt{\frac{y}{h}}\,dy$$

(12) $$I_x = b\int_0^h y^2\,dy - \frac{b}{\sqrt{h}}\int_0^h y^{2.5}\,dy$$

(13) $$= b\,\frac{1}{3}y^3\Big|_0^h - \frac{b}{\sqrt{h}}\,\frac{1}{3.5}\,y^{3.5}\Big|_0^h$$

(14) $$= \frac{bh^3}{3} - \frac{b}{\sqrt{h}}\,\frac{1}{3.5}\,h^{3.5}$$

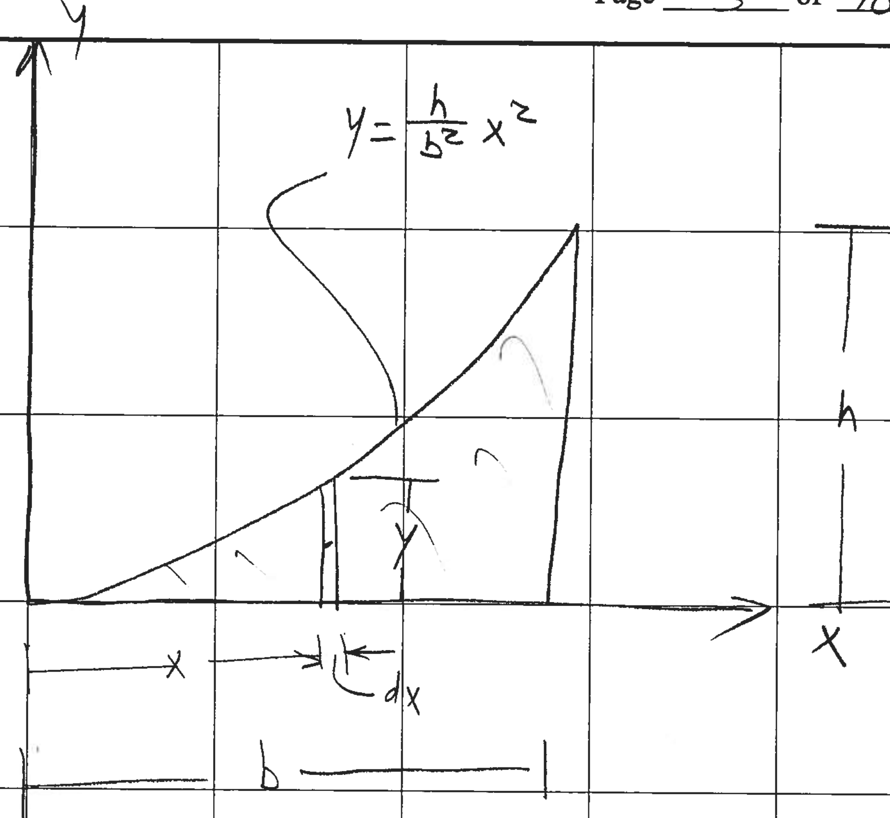
(15) $$= \frac{bh^3}{3} - \frac{bh^3}{3.5}$$

(16) $$= \frac{7bh^3}{21} - \frac{6bh^3}{21}$$

(17) $$\underline{I_x = \frac{bh^3}{21}}$$

<u>Calculate I_x with alternate method</u>

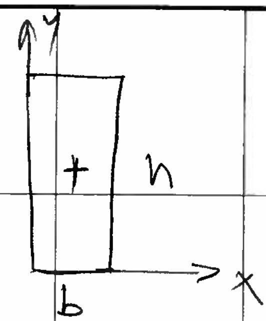
This method will use a vertical diff. element instead of the horizontal diff element as shown



Because the differential element is not parallel to the x-axis then we have to first express the moment of inertia of the differential element about the x-axis.

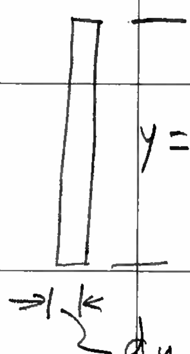
It is observed that the base of the differential element touches the x-axis for the entire area.

The moment of inertia of a rectangular area about its own base is given as follows.



$$I_x = \frac{bh^3}{3} \qquad (18)$$

Therefore the moment of differential moment of inertia associated with the differential element is



$$y = \frac{h}{b^2} x^2$$

$$dI_x = \frac{1}{3}\left(\frac{h}{b^2} x^2\right)^3 dx \qquad (19)$$

$$dI_x = \frac{1}{3}\frac{h^3}{b^6} x^6 dx \qquad (20)$$

Exploded View of Differential Element

The total moment of inertia is then given by integrating the differential moment of inertia across the area

$$(21) \qquad I_x = \int_0^b dI_x$$

(22) $$I_x = \int_0^b \frac{1}{3}\frac{h^3}{b^6} x^6\, dx$$

(23) $$I_x = \frac{h^3}{3b^6}\int_0^b x^6\, dx$$

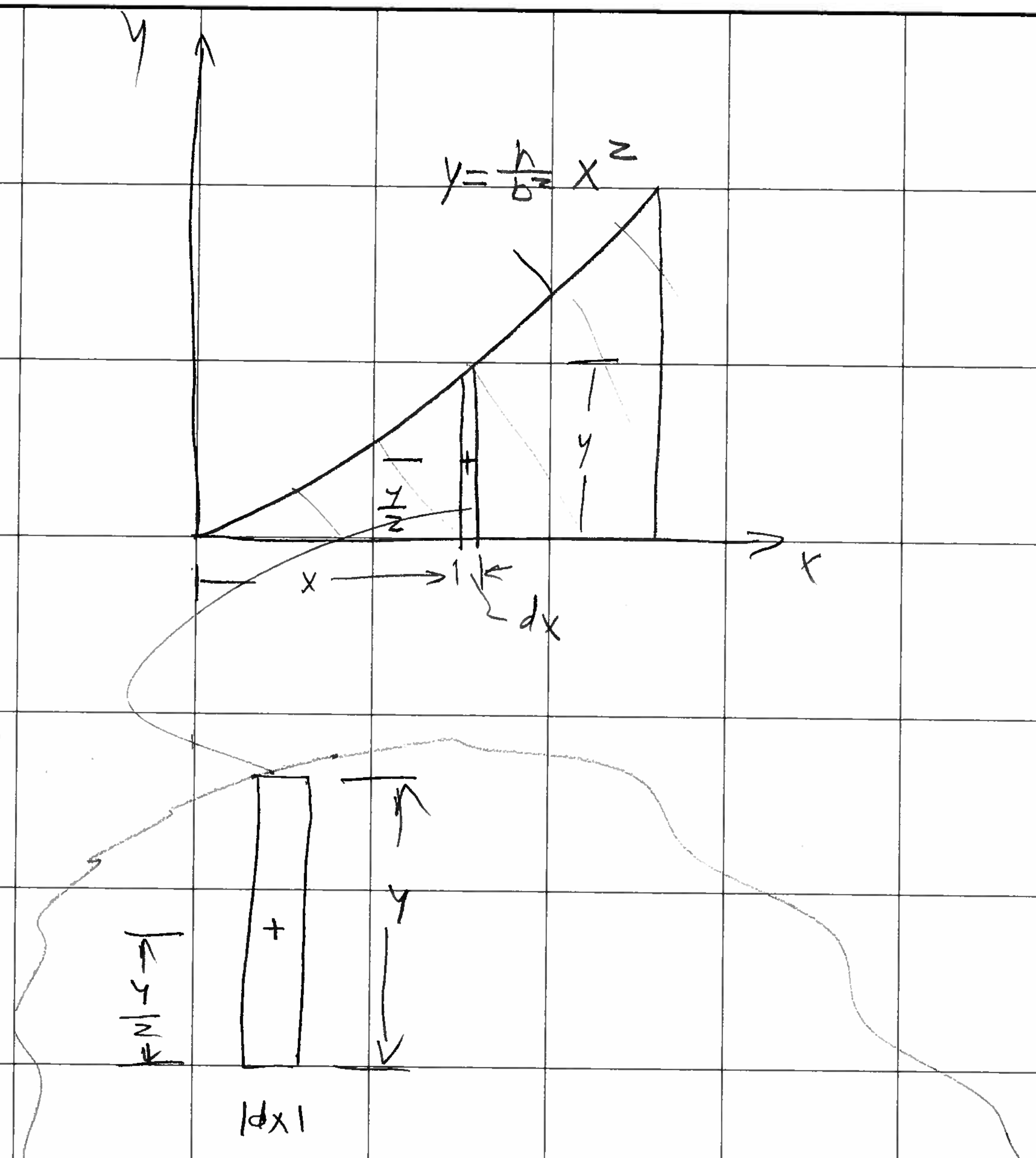
(24) $$I_x = \frac{h^3}{3b^6}\,\frac{1}{7}x^7\Big|_0^b$$

(25) $$I_x = \frac{h^3 b^7}{3b^6\, 7}$$

(26) $$\underline{I_x = \frac{b h^3}{21}}$$ (Same answer)

<u>Calculate I_x with another alternate method</u>

This method will use a vertical diff element as with the first alternate solution. However, advantage will not be taken of the fact that bottom of the differential element lines up with the x-axis. In this case the parallel axis theorem will have to be used to set up the expression for dI_x as follows.



Exploded View of differential element

The moment of inertia of the differential element about its own centroid is given as follows

(27) $$d\bar{I}_x = \frac{1}{12}(dx)\,y^3$$

(28) $$d\bar{I}_x = \frac{1}{12}\left(\frac{hx^2}{b^2}\right)^3 dx$$

(29) $$d\bar{I}_x = \frac{h^3x^6}{12b^6}\,dx$$

Then, the moment of inertia of the differential element about the x-axis is found using the parallel axis theorem as follows

(30) $$dI_x = d\bar{I}_x + \underbrace{dA}_{dA}\,d^2$$

(31) $$dI_x = \frac{h^3x^6}{12b^6}\,dx + (y\,dx)\left(\frac{y}{2}\right)^2$$

(32) $$dI_x = \frac{h^3x^6}{12b^6}\,dx + \left(\frac{h}{b^2}x^2\,dx\right)\left(\frac{h}{2b^2}x^2\right)^2$$

(33) $$dI_x = \frac{h^3x^6}{12b^6}\,dx + \frac{h^3x^6}{4b^6}\,dx$$

(34) $$dI_x = \frac{h^3x^6}{b^6}\left[\frac{1}{12} + \frac{1}{4}\right]dx$$

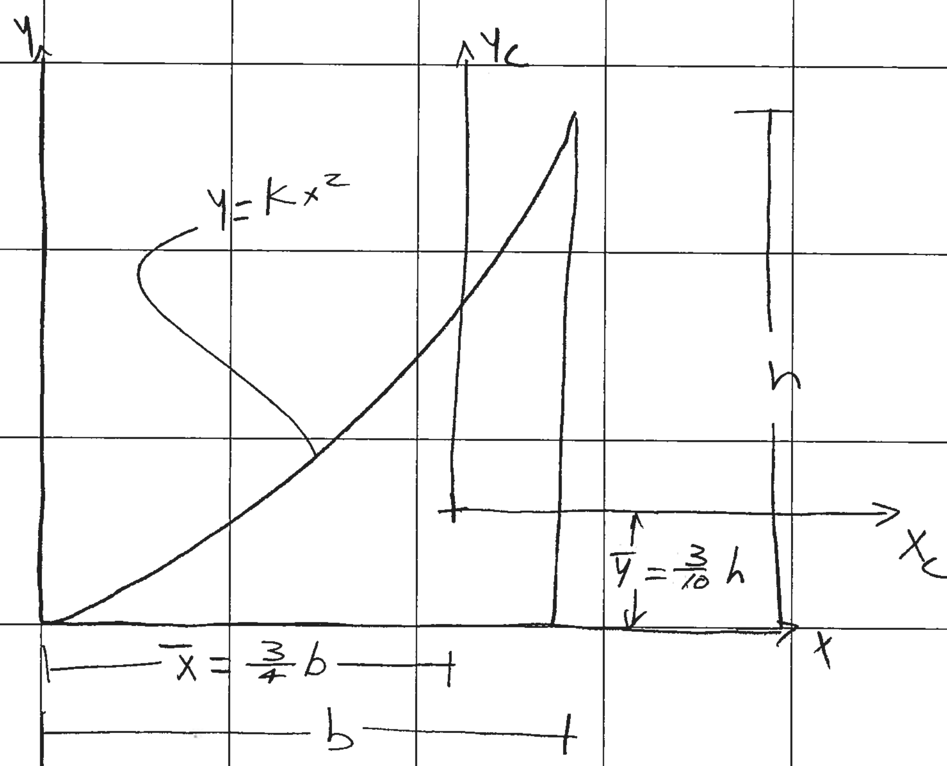
(35) $$\underline{dI_x = \frac{h^3x^6}{3b^6}}$$

The differential moment of inertia expressed in equation (35) is exactly the same as that expressed in equation (20). Thus the desired moment of inertia is as given in equation (26)

(36) $$I_x = \frac{bh^3}{21}$$

$$I_x = \frac{bh^3}{21} \quad \text{and} \quad I_y = \frac{hb^3}{5}$$

Calculate the moments of inertia for the shaded area shown about centroidal axes that are parallel to the x and y axes.



Given from previous example

$$\bar{X} = \frac{3}{4}b, \quad \bar{Y} = \frac{3}{10}h$$

$$A = \frac{bh}{3}, \quad I_x = \frac{bh^3}{21}, \quad I_y = \frac{hb^3}{5}$$

Calculate I_{x_c} using parallel axis theorem

(1) $$I_x = I_{x_c} + A d^2$$

(2) $$\frac{bh^3}{21} = I_{x_c} + \frac{bh}{3}\left(\frac{3}{10}h\right)^2$$

(3) $$I_{x_c} = \frac{bh^3}{21} - \frac{9\,bh^3}{300}$$

(4) $$I_{x_c} = \frac{bh^3}{(21)(300)}\left[300 - 9(21)\right]$$

(5) $$I_{x_c} = \frac{111\,bh^3}{6300}$$

(6) $$\underline{I_{x_c} = \frac{37}{2100}\,bh^3}$$

Calculate I_{y_c} using parallel axis theorem

(7) $$I_y = I_{y_c} + A d^2$$

(8) $$\frac{hb^3}{5} = I_{y_c} + \frac{bh}{3}\left(\frac{3}{4}b\right)^2$$

(9) $$I_{y_c} = \frac{hb^3}{5} - \frac{9}{48}hb^3$$

(10) $$I_{y_c} = \frac{hb^3}{240}\left[48 - 45\right]$$

(11) $$I_{y_c} = \frac{3hb^3}{240}$$

(12) $I_{y_c} = \dfrac{hb^3}{80}$